Digital and Analog Elements of Reality

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Abstract

This article shows what sort of discrete and analog elements are present in quantum mechanics itself. For this, we form quantum mechanics postulates connecting Stern-Gerlach, double-slit and EPR experiments. Which brings us to suggestion of how quantum and spacetime phenomena should be linked to settle a question about continuity of space, the major analog aspect of reality as we understand it today.

"Is Reality Digital or Analog?"

In approaching this broad question, let us clarify its scope, so that we can make brief arguments within allotted length for this essay.

When we use word *digital* to describe a device, it means that key functional aspects of a device are characterized by discrete variables that can be manipulated by discrete mathematics. And we use word *analog*, when key functional aspects of a device are characterized by continuous variables that can be manipulated by calculus. In the same manner, we will be able to say whether reality is digital or that reality is analog, if all fundamental aspects of reality are characterized either by discrete or by analog variables, respectively.

Let's then begin our survey of fundamental elements of reality, starting with matter.

1 Spin of fundamental particles

Fundamental particles, like electron and photons, have spin. Spin is an intrinsic characteristic applicable to any fundamental particle. Value of spin is $s = n/2$, where n is a non-negative integer. Hence, allowed values are $s = 0, 1/2, 1, \ldots$.

Standard Model tells us a story of what sort of particles are there in reality. According to it, there is a bunch of fermions, like electron, that have spin $s = 1/2$. There is a bunch of bosons, like photon, that have spin $s = 1$. And LHC is on the hunt for predicted Higgs boson, that has spin $s = 0$.

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To describe spin, we use discrete numbers. Therefore, phenomena of spin can be said to be of *digital* nature. But does everything related to spin have to be *digital*?

2 Stern-Gerlach experiment

Spin phenomena is commonly demonstrated by Stern-Gerlach experiment.

This experiment is arranged as follows. There is an electron gun that fires electrons with the same speed, in the same direction. Velocity (speed and direction) is the only set characteristic of produced electrons. Other characteristics like spin can be anything.

Electrons are fired and will follow a straight line, if unimpeded. When a magnetic field is set in electrons' path, electrons are deflected from a straight line, and follow two distinct trajectories.

Relation of these trajectories to spin can be explained in two ways. One is a usual description from a textbook, or from a provided above link. Usual description goes in step with a *historical* development of theories. The other way to describe what happens with electron in the magnetic field, is in terms of *current* fundamental theories. And at least out of curiosity, let us follow the second, less travelled route.

Quantum Electrodynamics (QED for short), is a part of Standard Model, and it describes electrons and photons. According to QED, magnetic field in Stern-Gerlach experiment, is a sort of sea of photons. Electron's interaction with a magnetic field is described by "bounces" with photons. These bounces are characterized in QED by a vertex at which photon is either absorbed or emitted by an electron.

Both electrons and photons are represented in QED by mathematical objects called spinors, that account for spin states. Spinors of both electron and photon "meet" at interaction vertex, which makes spins of both electron and photon coupled, or correlated. And, with an arrangement of photons into a magnetic field, this correlation influences a direction of electron's motion depending on how its spin is correlated with spin of photons in the field.

When an electron gun shoots electrons with *any* spin (denoted as $|?\rangle_{el}$) into a field of photons with some particular spin ($|x\rangle_{ph}$), electrons split into exactly two groups, each having a particular spin correlation with photons. These correlations can be called spin-up ($|x\uparrow\rangle_{el}$) and spin-down ($|x\downarrow\rangle_{el}$) relative to given photon field ($|x\rangle_{ph}$).

Keeping a record of number of electrons, we can write a process for field $|1\rangle_{ph}$ as

$$(N \;\; of \;\; |?\rangle_{el}) \xrightarrow{|1\rangle_{ph}} (N_{1\uparrow} \;\; of \;\; |1\uparrow\rangle_{el}) \;\; and \;\; (N_{1\downarrow} \;\; of \;\; |1\downarrow\rangle_{el})\,, \tag{1}$$

$$where \;\; N = N_{1\uparrow} + N_{1\downarrow} \tag{2}$$

If we send only $|1\uparrow\rangle_{el}$ electrons into photon sea $|2\rangle_{ph}$ that is the same as the original one $|1\rangle_{ph}$, we'll get only one group of electrons, and it will be again a spin-up group $|2\uparrow\rangle_{el}$. Which means that an electron with a spin-up relation to a particular photon will show the same spin-up relation to a photon that is similar to the initial one. This also implies

$|2\uparrow\rangle_{el} = |1\uparrow\rangle_{el}$. Altogether, we have

$$(N \ \ of \ \ \ |1\uparrow\rangle_{el}) \overset{|2\rangle_{ph}=|1\rangle_{ph}}{\longrightarrow} (N \ \ of \ \ \ |2\uparrow\rangle_{el}) = (N \ \ of \ \ \ |1\uparrow\rangle_{el}) \tag{3}$$

But when we change a photon sea to $|x\rangle_{ph} \neq |1\rangle_{ph}$, we will change its orientation relative to initial electrons. And electrons $|1\uparrow\rangle_{el}$ in process of interaction with $|x\rangle_{ph}$, will split again into two groups:

$$(N \ \ of \ \ \ |1\uparrow\rangle_{el}) \overset{|x\rangle_{ph}\neq|1\rangle_{ph}}{\longrightarrow} (N_{x\uparrow} \ \ of \ \ \ |x\uparrow\rangle_{el}) \ \ and \ \ (N_{x\downarrow} \ \ of \ \ \ |x\downarrow\rangle_{el}), \tag{4}$$

$$where \ \ N = N_{x\uparrow} + N_{x\downarrow} \tag{5}$$

Change of $|x\rangle_{ph}$ on experiment is done by way of rotating a magnetic field. And when it is rotated, trajectories of $|x\downarrow\rangle_{el}$ and $|x\uparrow\rangle_{el}$ also rotate relative to the lab, but stay at exactly same locations relative to the magnetic field. From this experimental observation we conclude that $|x\downarrow\rangle_{el}$ and $|x\uparrow\rangle_{el}$ are completely defined by $|x\rangle_{ph}$. Therefore, the only way to characterize orientation between $|x\rangle_{ph}$ and initial $|1\uparrow\rangle_{el}$ in (4) is by probabilities of electron getting from an initial state into allowed states after the correlation:

$$P_{1\uparrow\to x\uparrow} = \frac{N_{x\uparrow}}{N} \ \ and \ \ P_{1\uparrow\to x\downarrow} = \frac{N_{x\downarrow}}{N}, \tag{6}$$

$$P_{1\uparrow\to x\uparrow} + P_{1\uparrow\to x\downarrow} = 1 \tag{7}$$

Experiment shows that repeating a process, with the same relative orientation of $|x\rangle_{ph}$ and $|1\uparrow\rangle_{el}$, will yield the same values for probabilities $P_{1\uparrow\to x\downarrow}$ and $P_{1\uparrow\to x\downarrow}$. Therefore, these probabilities characterize allowed by nature orientation between $|x\rangle_{ph}$ and $|1\uparrow\rangle_{el}$, or between $|1\uparrow\rangle_{el}$ and a pair $|x\downarrow\rangle_{el}$, $|x\uparrow\rangle_{el}$, since this pair is completely defined by $|x\rangle_{ph}$.

With the notion of probabilities we may write (4) a bit more generally:

$$|\psi\rangle_{el} \overset{|x\rangle_{ph}}{\longrightarrow} (|x\uparrow\rangle_{el} \ \ with \ \ P_{\psi\to x\uparrow}) \ \ or \ \ (|x\downarrow\rangle_{el} \ \ with \ \ P_{\psi\to x\downarrow}), \tag{8}$$

$$P_{\psi\to x\uparrow} + P_{\psi\to x\downarrow} = 1 \tag{9}$$

where degenerate cases like (3) are realized by

$$P_{\psi\to x\uparrow} = 1, \ \ and \ \ P_{\psi\to x\downarrow} = 0 \ \ for \ \ |\psi\rangle_{el} = |x\uparrow\rangle_{el} \tag{10}$$

$$P_{\psi\to x\uparrow} = 0, \ \ and \ \ P_{\psi\to x\downarrow} = 1, \ \ for \ \ |\psi\rangle_{el} = |x\downarrow\rangle_{el} \tag{11}$$

3 Hilbert space

Elements $|\psi\rangle_{el}$, $|x\uparrow\rangle_{el}$ and $|x\downarrow\rangle_{el}$ are all states of electron, and can be thought as belonging to one set of mathematical entities. Probabilities, as mathematical entities, belong to numbers. To bring these two different mathematical things together into one structure, we can introduce certain operations over the set of electron states and numbers.

Working with statement (8), we may introduce a, so called, inner product between elements of electron states set from both sides of a statement, denoted as $\langle\psi|x\uparrow\rangle_{el}$ and $\langle\psi|x\downarrow\rangle_{el}$, which produces probability of getting from initial to final states as follows:

$$\langle\psi|x\uparrow\rangle_{el} \sim P_{\psi\to x\uparrow} \tag{12}$$
$$\langle\psi|x\downarrow\rangle_{el} \sim P_{\psi\to x\downarrow} \tag{13}$$

with degenerate cases like (10) leading to

$$\langle x\uparrow|x\downarrow\rangle_{el} = \langle x\downarrow|x\uparrow\rangle_{el} \sim 0 \tag{14}$$
$$\langle x\uparrow|x\uparrow\rangle_{el} = \langle x\downarrow|x\downarrow\rangle_{el} \sim 1 \tag{15}$$

Relations (14 – 15) look like scalar products between basis vectors in Euclidean vector space. Similarly to our inner product, a scalar product in Euclidean space takes two vectors from a set of vectors, and assigns a certain number to each pair.

Following a hunch that a relation between electron states may have a structure similar to Euclidean vector space, we introduce axioms that are similar to those in Euclidean vector space, which is an example of Hilbert space:

- Inner product between elements of the set gives a complex number[1]:
  $\langle\phi|\psi\rangle = a$
- Change of order in the product gives a complex conjugate result:
  $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* = a^*$
- Linearity, which will later allow to express any vector in terms of basis vectors, using a set of numbers, called coordinates:
  $\langle\phi|a\cdot\psi + b\cdot\upsilon\rangle = a\,\langle\phi|\psi\rangle + b\,\langle\phi|\upsilon\rangle$
  if we apply a complex conjugate to both sides and use a previous axiom, we get
  $\langle a\cdot\psi + b\cdot\upsilon|\phi\rangle = a^*\,\langle\psi|\phi\rangle + b^*\,\langle\upsilon|\phi\rangle$
- Product of any vector with itself $\langle\phi|\phi\rangle = \langle\phi|\phi\rangle^* = a = a^*$ is a number that is equal to its complex conjugate, making it a real number. So, we may add a condition applicable to real numbers:
  $\langle\phi|\phi\rangle \geq 0$

Armed with these axioms, we can give some physical sense to a newly constructed Hilbert space. $\langle\psi|x\uparrow\rangle_{el}$ is some complex number, while $P_{\psi\to x\uparrow}$ is a real number, and to connect the two, like in (12 – 13), we postulate:

$$\|\langle\psi|x\uparrow\rangle_{el}\|^2 = \|c_{\psi\to x\uparrow}\|^2 = c_{\psi\to x\uparrow}c^*_{\psi\to x\uparrow} = P_{\psi\to x\uparrow} \tag{16}$$
$$\|\langle\psi|x\downarrow\rangle_{el}\|^2 = \|c_{\psi\to x\downarrow}\|^2 = c_{\psi\to x\downarrow}c^*_{\psi\to x\downarrow} = P_{\psi\to x\downarrow} \tag{17}$$

[1] There is absolutely no significance in having complex numbers, except that real numbers will not satisfy some requirements and uses of a constructed space. It is similar to the mathematical fact that a quadratic equation always has solutions in complex numbers, of which a set of real numbers is just a subset.

Relations (14 – 15) transform into equalities

$$\langle x\uparrow | x\downarrow\rangle_{el} = \langle x\downarrow | x\uparrow\rangle_{el} = 0 \tag{18}$$
$$\langle x\uparrow | x\uparrow\rangle_{el} = \langle x\downarrow | x\downarrow\rangle_{el} = 1 \tag{19}$$

On experiment, for any electron $|\psi\rangle_{el}$, a photon field configuration $|x\rangle_{ph}$ can be found, such that $|\psi\rangle_{el} = |x\uparrow\rangle_{el}$ (a degenerate case). Then (19) forces

$$\langle\psi|\psi\rangle_{el} = 1, \tag{20}$$

which is usually referred as normalization requirement. Product is connected to probabilities, and probabilities should add up to 1. This leads to normalization requirement.

Let's do the following manipulation with (20) and (9):

$$\begin{aligned}\langle\psi|\psi\rangle_{el} &= 1 = P_{\psi\to x\uparrow} + P_{\psi\to x\downarrow} \\ &= c^*_{\psi\to x\uparrow} c_{\psi\to x\uparrow} + c^*_{\psi\to x\downarrow} c_{\psi\to x\downarrow} \\ &= c^*_{\psi\to x\uparrow} \langle\psi|x\uparrow\rangle_{el} + c^*_{\psi\to x\downarrow} \langle\psi|x\downarrow\rangle_{el}\end{aligned}$$

since this must hold for any $|\psi\rangle_{el}$, we may write

$$|\psi\rangle_{el} = c^*_{\psi\to x\uparrow} |x\uparrow\rangle_{el} + c^*_{\psi\to x\downarrow} |x\downarrow\rangle_{el} \tag{21}$$

From a purely mathematical side, decomposition (21) applies to all vectors in a space because (18 – 19) provide a basis in a constructed space. And mathematical statements (21) and (20) lead to correct sum of probabilities (9). It is a sort of opposite derivation that simply highlights mathematical consistency.

We have constructed a Hilbert space to characterize initial electron states $|\psi\rangle_{el}$ in terms of final electron states $|x\uparrow\rangle_{el}$ and $|x\downarrow\rangle_{el}$ that occur before and after the interaction with a photon field in Stern-Gerlach experiment.

Allowed relative spin correlations are expressed by basis vectors like $|x\uparrow\rangle_{el}$ and $|x\downarrow\rangle_{el}$, ant their number is countable. Therefore, relative spin correlation is a digital phenomena.

Relative initial spin orientations between electrons $|\psi\rangle_{el}$ and photon field $|x\rangle_{ph}$ are captured by our Hilbert space (cause $|x\uparrow\rangle_{el}$ and $|x\downarrow\rangle_{el}$ are defined by $|x\rangle_{ph}$). And since our space is complete, a number of allowed relative spin orientations before interaction between an electron and photon field is uncountably large, requiring continuum treatment. Therefore, relative spin orientation is an analog phenomena.

At this point we should recall that a change in relative spin orientation between an electron and photons in field is done by rotation of a magnetic field in our three dimensional space. Which lands two questions. First, do spacetime properties contribute to our constructed space. Second, will we have to make our constructed space smaller (make it incomplete in mathematical sense), if spacetime is found to be discrete. Depending on answers to these questions, we may have to change our assertion about analog nature of relative spin orientation. And, in order to raise questions about spacetime, we should develop some more concepts about quantum mechanical phenomena.

4 Quantum Mechanics postulates

4.1 Postulate I. Probabilistic nature of interaction

In the description of Stern-Gerlach experiment we cheated a little bit. We were talking about electron only. We constructed a space of electron states. But we did not mention that interaction between photon field and electron influences both. In particular, when an electron changes its path from a straight line, it changes its momentum. As a result, photons of a magnetic field take this little hit of momenta. And the whole interaction, depicted by (8), should really look more symmetric:

$$\begin{aligned} |\psi\rangle_{el}\,, |x\rangle_{ph} \quad \longrightarrow \quad & \Big(|x\uparrow\rangle_{el}\,, |x_{\uparrow}\rangle_{ph} \quad with \quad P^{el}_{\psi \to x\uparrow} = P^{ph}_{x \to x_{\uparrow}} \Big) \\ or \quad & \Big(|x\downarrow\rangle_{el}\,, |x_{\downarrow}\rangle_{ph} \quad with \quad P^{el}_{\psi \to x\downarrow} = P^{ph}_{x \to x_{\downarrow}} \Big) \end{aligned}$$

By the same token, to make a general postulate about nature of interaction, we should incorporate all related sides.

> **Postulate I:** When two systems ψ and ϕ interact with each other, they undergo a transition from some initial states $|\psi_{init}\rangle$ and $|\phi_{init}\rangle$ to final states $|\psi_j\rangle$ and $|\phi_j\rangle$, where only one of j possibilities may occur at a time with some probability:
>
> $$|\psi_{init}\rangle\,, |\phi_{init}\rangle \longrightarrow \bigcup_j \Big(|\psi_j\rangle\,, |\phi_j\rangle \quad with \quad P^{\psi}_{\psi_{init} \to \psi_j} = P^{\phi}_{\phi_{init} \to \phi_j} \Big)\,, \tag{22}$$
>
> $$where \quad \sum_j P^{\psi}_{\psi_{init} \to \psi_j} = \sum_j P^{\phi}_{\phi_{init} \to \phi_j} = 1 \tag{23}$$
>
> We will say that states $|\psi_j\rangle$ and $|\phi_j\rangle$ in each final pair are entangled with each other as a result of interaction.

We should be able to construct state spaces, as we did in section 3, for both ψ and ϕ. And when we do, we should fit these together, so that *a)* probabilities calculated on both sides are equal for the same pair of final states, *b)* probabilities calculated on each of the side add to one, and *c)* like in (18 – 19), final j vectors should be mutually orthogonal, and respective *init* vectors should be decomposable in terms of these j vectors, similar to (21).

In section 3 we have constructed a space of electron states. But we haven't done so for photons in a magnetic field, that is generated by magnets, which are powered from electric grid, etc. Photons in a Stern-Gerlach experiment are part of a complex system, and this necessarily includes aspects addressed in the next postulate.

4.2 Postulate II. Interaction confinement

Postulate I tells us what happens when two systems ψ and ϕ interact. But the world does not consists of just two systems. Let's then ask, what will happen to the process (22), if we add some system Υ, which does not interact with neither system ψ, nor system ϕ.

If nothing happens, then we will rewrite (22) as

$$\left|\Upsilon\right\rangle\left(\left|\psi_{init}\right\rangle,\left|\phi_{init}\right\rangle\right) \longrightarrow \left|\Upsilon\right\rangle\left(\bigcup_{j}\left(\left|\psi_j\right\rangle,\left|\phi_j\right\rangle \quad with \quad P_j\right)\right) \tag{24}$$

But what can possibly happen? Some philosophical considerations, based on everyday experience of a classical world, may lead one to suggest that a choice of final states in interaction of ψ and ϕ should be known to the rest of the world, i.e. to any other system like Υ, even if it is not interacting with neither ψ nor ϕ. This suggestion can be depicted as

$$\left|\Upsilon\right\rangle\left(\left|\psi_{init}\right\rangle,\left|\phi_{init}\right\rangle\right) \longrightarrow \bigcup_{j}\left(\left|\Upsilon\right\rangle\left(\left|\psi_j\right\rangle,\left|\phi_j\right\rangle\right) \quad with \quad P_j\right) \tag{25}$$

In the case of (25), any unsuspecting system, like Υ, gets entangled with final states of interaction that happen between ψ and ϕ. Entanglement instantaneously spreads to the rest of the world, without any regard that a system like Υ may be light-years away from ψ and ϕ.

As nonlocal as option (25) sounds, the first option (24) may not be easy to swallow, either. It says that any choice of final states shall stay confined to interacting systems. A composite system (ψ, ϕ) does change its state as a result of internal interaction, but no entanglement choices "leak outside" when there is no interaction with the rest of the world, i.e. when (ψ, ϕ) is a closed system.

Can experimental evidence tell us which option nature itself prefers?

Let's recall double-slit experiment. And let's say that electrons are thrown one by one at slits, with a subsequent interference pattern seen on the screen. In it, electron together with slits, or rather material in which slits are cut, form a composite system like (ψ, ϕ). Now, if (25) is correct, then a choice of a slit, through which electron passes, should be known to the rest of the world, i.e. we should be entangled with this choice, even without any interaction with the system. Then a little electron counting, done at each slit, should make no difference, as it will only confirm an already known choice. Yet, it does, according to an experiment. Therefore, option (25) is incorrect, and, by implication, option (24) is preferred by nature.

> **Postulate II:** When interaction occurs between subsystems of a closed system, resulting entanglements between final states and choices thereof are confined to the system.

In (24) composite system (ψ, ϕ) is "not loosing" any probability, as far as external system Υ is concerned. Therefore, a state vector that describes closed system (ψ, ϕ) should keep the same unit length as it goes through its evolution. In other words, evolution of a closed system is described by unitary transformation.

On a bit technical note, the following, dependent on a parameter t, operator $\hat{U}$ is unitary, when $\hat{H}$ is hermitian:

$$\hat{U} = exp\left(-\frac{i}{\hbar}\hat{H}t\right) \tag{26}$$

It may describe a transformation of a state vector as a function of parameter t:

$$|\psi(t)\rangle = \hat{U}\,|\psi(0)\rangle$$

Taking a t derivative will do the following:

$$\begin{aligned}\frac{\partial}{\partial t}|\psi(t)\rangle &= \frac{\partial}{\partial t}\hat{U}\,|\psi(0)\rangle = \frac{\partial}{\partial t}exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\psi(0)\rangle \\ &= -\frac{i}{\hbar}\hat{H}exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\psi(0)\rangle = -\frac{i}{\hbar}\hat{H}\hat{U}\,|\psi(0)\rangle = -\frac{i}{\hbar}\hat{H}\,|\psi(t)\rangle\end{aligned}$$

Which turns out to be a Schrodinger equation with time-independent Hamiltonian $\hat{H}$:

$$\hat{H}\,|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \tag{27}$$

To be mathematically clean, a Schrodinger equation should actually be postulated. Yet, above derivation shows that *a)* any further postulated or suggested evolution equation(s) must be compatible with Postulate II by providing unitarity, *b)* infinitesimal change in time is described by energy operator, and *c)* time comes into quantum mechanics as a mere parameter that labels order of states.

5 Quantum correlations and spacetime

We have used Stern-Gerlach experiment to motivate Postulate I, and we used double-slit experiment to formulate Postulate II. Yet, there is one more experiment that exposes "weirdness" of quantum nature. Let's consider here a situation, which is also used in EPR paradox and in a practical experiment to test Bell's theorem.

Two electrons are created in some interaction with opposite to each other spins. Details of an interaction are not important, except that it obeys Postulate II.

Out of two possible correlations between electrons, only one with opposing spins is allowed, and, from internal point of view, it happens with certainty.

Yet, relative to the rest of the world, or the lab, in particular, this system of two electrons as a whole should have spin 0. Then a question arises, what will be seen, when we try to determine spins of each particular electron.

To determine electrons' spins, we can use Stern-Gerlach apparatus. It should be aligned in some way, but there is no preferred orientation at all, cause a system of both electrons has spin 0 as a whole.

So, on experiment when the first electron is measured against some randomly chosen orientation z, it produces spin-up $|z\uparrow\rangle$ and spin-down $|z\downarrow\rangle$ results with 50% probability. When the second electron is tested, it produces for the same orientation exactly opposite results with certainty (probability 100%), which is fully compatible with the initial requirement that spins are opposite. Also 100% probability is attained only because a correlation between the first electron and the lab sets a preferred orientation between previously closed system of electrons and the lab, relative to which both spins are aligned in opposite directions.

This sounds perfect till we introduce an idea that all matter states and all interactions happen *inside* relativistic spacetime, and, given enough spatial separation, it is impossible to tell which of electrons' measurement was the first to introduce preferred direction.

In this light, instead of saying that stuff happens *in* time and *in* space, may be, we should *define* space and time *in terms of* matter states and interactions between them.

This is a relational approach to spacetime, which suggests that spacetime *is* a network of relations between matter elements. And it can possibly be the right approach to bring together quantum and spacetime descriptions in a consistent way, which may as well settle the question of whether reality is ultimately digital or analog.